Contracted and expanded integrable structures

2009 J. Phys. A: Math. Theor. 42 475204

(http://iopscience.iop.org/1751-8121/42/47/475204)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 475204 (17pp) doi:10.1088/1751-8113/42/47/475204

Contracted and expanded integrable structures

Anastasia Doikou and Konstadinos Sfetsos

Department of Engineering Sciences, University of Patras, 26110 Patras, Greece

E-mail: adoikou@upatras.gr and sfetsos@upatras.gr

Received 27 May 2009, in final form 29 September 2009
Published 4 November 2009
Online at stacks.iop.org/JPhysA/42/475204

Abstract
We propose a generic framework to obtain certain types of contracted and centrally extended algebras. This is based on the existence of quadratic algebras (reflection algebras and twisted Yangians), naturally arising in the context of boundary integrable models. A quite old misconception regarding the 'expansion' of the E_2 algebra into sl_2 is resolved using the representation theory of the aforementioned quadratic algebras. We also obtain centrally extended algebras associated with rational and trigonometric (q-deformed) R-matrices that are solutions of the Yang–Baxter equation.

PACS number: 02.20.Uw

1. Introduction

The original motivation for the present work comes from the following interesting property of the non-semi-simple algebras that one obtains under Inonü–Wigner-type contractions. It looks as if there is a nonlinear realization of the original semi-simple algebra in terms of the generators of the non-semi-simple algebras, which comes under the general name of expansion (see for instance [1]). As a concrete example consider the E_2 algebra corresponding to the three-dimensional Euclidean group. This arises also as an Inonü–Wigner contraction of the sl_2 and $su(2)$ algebras. The commutation rules are

$$[J, P^{\pm}] = \pm P^{\pm}, \qquad [P^+, P^-] = 0, \tag{1.1}$$

with $C = P^+P^-$ being the quadratic Casimir operator. Let us define

$$Y^{\pm} = J P^{\pm}. \tag{1.2}$$

Then, it is straightforward to show that

$$[J, Y^{\pm}] = \pm Y^{\pm}, \qquad [Y^+, Y^-] = -2J P^+P^-, \tag{1.3}$$

which are essentially the commutation relations of the sl_2 algebra. Defining $\tilde{Y}^{\pm} = \frac{Y^{\pm}}{\sqrt{P^+P^-}}$, then the elements $\tilde{Y}^{\pm}, J$ seem to generate the sl_2 algebra. A similar realization leads to the $su(2)$ commutation relations, that is $\tilde{Y}^{\pm} = \frac{Y^{\pm}}{\sqrt{-P^{+}P^{-}}}$. The result is very appealing but it has a few drawbacks. From a physical point of view, it is very difficult to visualize how information defined in the two-dimensional infinite flat plane can be used to reconstruct the curved manifold corresponding to the group spaces for $SU(2)$ and $SL(2)$. In addition, from a more mathematical view point we can readily check that it is only the identity of the corresponding Lie algebras that can be expressed in terms of E_2 representations.

We will show that this so-called expansion is only an apparent one. In particular, the basic difference is already encoded from the algebraic point of view in the co-product. For the generators of sl_2 this is given by

$$\Delta(X) = \mathbb{I} \otimes X + X \otimes \mathbb{I}, \qquad X \in \{J, J^{\pm}\}. \tag{1.4}$$

Then one naively computes that

$$\Delta(Y^{\pm}) = Y^{\pm} \otimes \mathbb{I} + \mathbb{I} \otimes Y^{\pm} + J \otimes P^{\pm} + P^{\pm} \otimes J, \tag{1.5}$$

which already suggests that, contrary to what one would have expected, $\Delta(Y^{\pm})$ and $\Delta(J)$ do not belong to $U(sl_2) \otimes U(sl_2)$—note in (1.5) one borrows elements from E_2 in order to construct the co-product—, although they still satisfy (1.3), so one should better search for a broader algebra. Indeed, we show here that the associated extended algebra in this case is the contracted sl_2 twisted Yangian (see e.g. [2–5]), which rules integrable models with non-trivial integrable boundary conditions. In other words, we use the notion of integrability in order to show that the 'expansion' presented in [1] is not valid.

Moreover, we show that the symmetry breaking mechanism due to the presence of appropriate boundary conditions may also be exploited in order to obtain centrally extended algebras via suitable contraction procedures. We also use the boundary algebra to obtain the relevant Casimir. This is perhaps the simplest and most straightforward means to obtain the Casimir of usual and deformed Lie algebras. One of the main points of this investigation is that we are able to show that the associated open transfer matrix commutes with the elements of the emerging contracted algebra. We study here the simplest case, that is the E_2^c algebra, in order to illustrate the procedure followed; however, this description may be generalized for more complicated algebraic structures. Such an exhaustive analysis however is beyond the intended scope of the present work.

Let us now outline the content of this paper. In section 2, we introduce the fundamental quadratic algebras that rule integrable models with periodic and non-trivial boundary conditions, that is we review relevant aspects of the Yang–Baxter equation [6] and the associated quadratic relations (see e.g. [7, 8]) which give sire to the quantum algebras [9–12]. In a similar spirit, we also review the reflection algebra and the twisted Yangian [2, 5, 13]. In section 3, we examine the contracted sl_2 twisted Yangian. We show, using representation theory of the relevant algebra, that the 'expansion' presented in the literature [1] is not sl_2, but the associated E_2 twisted Yangian. Next, we exploit symmetry breaking mechanisms due to the presence of suitable integrable boundary conditions in order to obtain the centrally extended E_2^c algebra. In section 5, we construct the q-deformed version of the E_2^c algebra with the help of $U_q(sl_2) \otimes u(1)$ boundary symmetry. In the last section, a discussion on possible directions for further study is presented.

2. Quadratic algebras

In this section, we give a short review of the fundamental quadratic algebraic relations, ruling the quantum integrable models, that is the Yang–Baxter and reflection equations.

The Yang–Baxter equation [6] is defined as

$$R_{12}(\lambda_1-\lambda_2)R_{13}(\lambda_1)R_{23}(\lambda_2)=R_{23}(\lambda_2)R_{13}(\lambda_1)R_{12}(\lambda_1-\lambda_2), \tag{2.1}$$

acting on $\mathbb{V}^{\otimes 3}$, and $R\in \mathrm{End}(\mathbb{V}^{\otimes 2})$ $R_{12}=R\otimes\mathbb{I}$, $R_{23}=\mathbb{I}\otimes R$. From a physical point of view, it is well known that the Yang–Baxter equation describes the factorization of multi-particle scattering in integrable models (see e.g. [14–17]). Given an R matrix, as a solution of the Yang–Baxter equation, we introduce the following fundamental algebraic relations [7], which essentially define a particular algebra $\mathcal{A}$ (see e.g. [7]),

$$R_{12}(\lambda_1-\lambda_2)L_1(\lambda_1)L_2(\lambda_2)=L_2(\lambda_2)L_1(\lambda_1)R_{12}(\lambda_1-\lambda_2), \tag{2.2}$$

where $L\in\mathrm{End}(\mathbb{V})\otimes\mathcal{A}$, with $\mathcal{A}$ being the algebra defined by (2.2). This allows the construction of tensorial representations of the later algebra as

$$T_a(\lambda)=L_{aN}(\lambda-\theta_N)L_{aN-1}(\lambda-\theta_{N-1})\cdots L_{a2}(\lambda-\theta_2)L_{a1}(\lambda-\theta_1), \tag{2.3}$$

where $T(\lambda)\in\mathrm{End}(\mathbb{V})\otimes\mathcal{A}^{\otimes N}$. For historical reasons, the a space is called 'auxiliary', whereas the spaces $1,\ldots,N$ are called 'quantum'. For simplicity, we usually suppress all quantum spaces when writing down the monodromy matrix. Also θ_i are free complex parameters and are called inhomogeneities. Using the fundamental algebra (2.2) one may show that

$$[\mathrm{tr}\,T(\lambda),\mathrm{tr}\,T(\mu)]=0, \tag{2.4}$$

where $\mathrm{tr}\,T(\lambda)\in\mathcal{A}^{\otimes N}$ and the trace is taken over the auxiliary space. The latter relation guarantees the integrability of the system. Once the algebra $\mathcal{A}$ is represented, the tensorial representation acquires the meaning of the monodromy matrix of a quantum spin chain and $\mathrm{tr}\,T$, the corresponding transfer matrix, may be diagonalized using for instance Bethe ansatz techniques.

We shall now introduce the reflection equation [2, 13]. In fact, we shall consider two types of equations associated with two distinct algebras: the reflection algebra [2] and the twisted Yangian [3, 5]. These two algebras describe essentially the algebraic content of integrable models with two distinct types of boundary conditions known as soliton preserving (SP) and soliton non-preserving (SNP), respectively. A spin chain-like system with SNP boundary conditions was first derived in [4], whereas generalizations were studied in [18, 19].

Reflection algebra. The equation associated with the reflection algebra $\mathcal{R}$ (SP boundary conditions) is given by [13, 2]

$$R_{12}(\lambda_1-\lambda_2)\mathbb{K}_1(\lambda_1)R_{21}(\lambda_1+\lambda_2)\mathbb{K}_2(\lambda_2)=\mathbb{K}_2(\lambda_2)R_{12}(\lambda_1+\lambda_2)\mathbb{K}_1(\lambda_1)R_{21}(\lambda_1-\lambda_2), \tag{2.5}$$

acting on $\mathbb{V}^{\otimes 2}$ and as customary we follow the notation $\mathbb{K}_1=\mathbb{K}\otimes\mathbb{I}$ and $\mathbb{K}_2=\mathbb{I}\otimes\mathbb{K}$. Also $R_{21}=\mathcal{P}R_{12}\mathcal{P}$, where $\mathcal{P}$ is the permutation operator: $\mathcal{P}(a\otimes b)=b\otimes a$ and also $\mathbb{K}\in\mathrm{End}(\mathbb{V})\otimes\mathcal{R}$.

Twisted Yangian. The twisted Yangian (SNP boundary conditions) $\mathcal{T}$ is defined by [3, 5]

$$R_{12}(\lambda_1-\lambda_2)\tilde{\mathbb{K}}_1(\lambda_1)\hat{R}_{12}(\lambda_1+\lambda_2)\tilde{\mathbb{K}}_2(\lambda_2)=\tilde{\mathbb{K}}_2(\lambda_2)\hat{R}_{12}(\lambda_1+\lambda_2)\tilde{\mathbb{K}}_1(\lambda_1)R_{12}(\lambda_1-\lambda_2), \tag{2.6}$$

where

$$\hat{R}_{12}(\lambda)=R_{12}^{t_1}(-\lambda-\mathrm{i}\rho), \tag{2.7}$$

with $\tilde{\mathbb{K}}\in\mathrm{End}(\mathbb{V})\otimes\mathcal{T}$ and ^{t_1} denotes transposition on the first space.

In general, the representations of the later algebras may be expressed, in an index free notation, as [2–4]

$$\begin{aligned}&\text{Reflection algebra:}\quad \mathbb{K}(\lambda)=L(\lambda-\Theta)K(\lambda)L^{-1}(-\lambda-\Theta),\\ &\text{Twisted Yangian:}\quad \tilde{\mathbb{K}}(\lambda)=L(\lambda-\Theta)\tilde{K}(\lambda)L^{t}(-\lambda-\rho-\Theta),\end{aligned} \tag{2.8}$$

where the matrices K, $\tilde{K}$ are c-number representations of the aforementioned algebras, Θ is an inhomogeneity and ρ is a constant that depends on the underlying algebra. For instance, in the gl_n case $\rho = \frac{n}{2}$. Tensor representations of these algebra are given by

$$\begin{aligned} &\text{Reflection algebra:} \quad \mathbb{T}_0(\lambda) = T_0(\lambda) K_0(\lambda) T_0^{-1}(-\lambda), \\ &\text{Twisted Yangian:} \quad \tilde{\mathbb{T}}_0(\lambda) = T_0(\lambda) \tilde{K}_0(\lambda) T_0^{t_0}(-\lambda - \mathrm{i}\rho), \end{aligned} \tag{2.9}$$

where we recall that T is defined in (2.3).

Let us now define the N 'particle' transfer matrix

$$t(\lambda) = \mathrm{tr}\{K^+(\lambda)\mathbb{T}(\lambda)\}, \qquad \tilde{t}(\lambda) = \mathrm{tr}\{\tilde{K}^+(\lambda)\tilde{\mathbb{T}}(\lambda)\}. \tag{2.10}$$

Clearly, for the one 'particle' construction we have $\mathbb{T} \to \mathbb{K}$ and $\tilde{\mathbb{T}} \to \tilde{\mathbb{K}}$. Also the K^+, $\tilde{K}^+$ matrices are c-number solutions of the reflection algebras and twisted Yangian, respectively. With the help of the quadratic exchange relations, one may show that (see e.g. [2, 4])

$$[t(\lambda), t(\mu)] = 0, \qquad [\tilde{t}(\lambda), \tilde{t}(\mu)] = 0. \tag{2.11}$$

In the following, we shall mainly consider the gl_n Yangian $\mathcal{Y}(gl_n)$ case. The associated R- and L-matrices are then given by [20]

$$R(\lambda) = \mathbb{I} + \frac{\mathrm{i}}{\lambda}\mathcal{P}, \qquad L(\lambda) = \mathbb{I} + \frac{\mathrm{i}}{\lambda}\mathbb{P}, \tag{2.12}$$

where the entries $\mathbb{P}_{ab} \in gl_n$.

3. sl_2 quadratic algebras; contractions and expansions

In this section, we shall focus on the situation associated with the $\mathcal{Y}(sl_2)R$-matrix [20]. Then the L-operator in (2.12) is explicitly given by

$$L(\lambda) = \mathbb{I} + \frac{\mathrm{i}}{\lambda}\begin{pmatrix} J + \frac{1}{2} & -J^- \\ J^+ & -J + \frac{1}{2} \end{pmatrix}, \tag{3.1}$$

with J, $J^\pm$ being the generators of sl_2 satisfying

$$[J, J^\pm] = \pm J^\pm, \qquad [J^+, J^-] = -2J. \tag{3.2}$$

Recall that the Casimir operator for sl_2 is $C = -J^2 + \frac{1}{2}(J^+J^- + J^-J^+)$.[1] As is well known, one may easily obtain from sl_2 by an Inonü–Wigner contraction the E_2 algebra. Indeed setting $J^\pm = \frac{1}{\epsilon}P^\pm$ with $\epsilon \to 0$ we have from (3.2) that

$$[J, P^\pm] = \pm P^\pm, \qquad [P^+, P^-] = 0. \tag{3.5}$$

The associated Casimir operator is $C = P_+P_-$.

Let us point out that in the sl_2 case the reflection algebra coincides essentially with the twisted Yangian due to the fact that sl_2 is self-conjugate. Equivalently, the sl_2 L-operator and the R-matrix are crossing symmetric, i.e.

$$\sigma^y L^t(-\lambda - \mathrm{i})\sigma^y = L(\lambda), \qquad \sigma_1^y R_{12}^{t_1}(-\lambda - \mathrm{i})\sigma_1^y = R_{12}(\lambda). \tag{3.6}$$

[1] Note that there exist a representation π and a homomorphism h defined as $\pi : su_2 \hookrightarrow \mathrm{End}(\mathbb{C}^2)$ and $h : su_2 \hookrightarrow sl_2$ such that

$$\pi(J) = \frac{\sigma^z}{2}, \qquad \pi(J^+) = \sigma^+, \qquad \pi(J^-) = \sigma^- \qquad h(J) = J, \qquad h(J^+) = J^+, \qquad h(J^-) = -J^-. \tag{3.3}$$

Taking into account the above we conclude that

$$(\pi \otimes h)\Delta(\mathrm{x})L(\lambda) = L(\lambda)(\pi \otimes h)\Delta(\mathrm{x}) \qquad \Delta(\mathrm{x}) = \mathbb{I} \otimes \mathrm{x} + \mathrm{x} \otimes \mathbb{I}, \qquad \mathrm{x} \in su_2. \tag{3.4}$$

Moreover, one can easily check that, due to the fact that $\mathbb{P}^2$ is proportional to the Casimir operator, we have

$$L^{-1}(-\lambda) \propto L(\lambda). \tag{3.7}$$

Taking into account the representations of the reflection algebra and the twisted Yangian and the above relations, it is straightforward to see that the two algebras coincide.

In general, the twisted Yangian can be defined up to a gauge transformation, that is $\hat{L} \to V\hat{L}V$, where $V^2 = 1$. Then the defining relations of the twisted Yangian are slightly modified after $\hat{R}$ has also been redefined. Indeed, by multiplying equation (2.6) from the right with V_1V_2, we end up with

$$R_{12}(\lambda_1-\lambda_2)\bar{\mathbb{K}}_1(\lambda_1)\bar{R}'_{12}(\lambda_1+\lambda_2)\bar{\mathbb{K}}_2(\lambda_2) = \bar{\mathbb{K}}_2(\lambda_2)\bar{R}_{12}(\lambda_1+\lambda_2)\bar{\mathbb{K}}_1(\lambda_1)R'_{12}(\lambda_1-\lambda_2), \tag{3.8}$$

where we define

$$\bar{\mathbb{K}} = \tilde{\mathbb{K}}V, \qquad \bar{R}_{12}(\lambda) = V_2R_{12}^{t_1}(-\lambda-\mathrm{i}\rho)V_2, \qquad A'_{12} = V_1V_2A_{12}V_1V_2. \tag{3.9}$$

A representation of the algebra defined in (3.8) is

$$\bar{\mathbb{K}} = L(\lambda-\Theta)\hat{L}(\lambda+\Theta), \qquad \text{where} \quad \hat{L}(\lambda) = VL^t(-\lambda-\mathrm{i}\rho)V. \tag{3.10}$$

For simplicity, we set henceforth $\Theta = \frac{\mathrm{i}}{2}$.

Although we have seen that in this case the reflection algebra and twisted Yangian coincide, we will preserve the terminology to basically distinguish two different types of boundary conditions, which are discussed below.

3.1. The twisted Yangian and its contraction

Here we shall consider[2]

$$\hat{L}(\lambda) = \sigma^x L^t(-\lambda-\mathrm{i})\sigma^x = 1 + \frac{\mathrm{i}}{\lambda}\begin{pmatrix} J+\frac{1}{2} & J^- \\ -J^+ & -J+\frac{1}{2}\end{pmatrix}, \tag{3.11}$$

where we have used (2.12) with (3.1). We form the generating function

$$\bar{\mathbb{K}}(\lambda) = L\left(\lambda-\frac{\mathrm{i}}{2}\right)\hat{L}\left(\lambda+\frac{\mathrm{i}}{2}\right), \tag{3.12}$$

where from (3.1) and (3.11)

$$\begin{aligned} L\left(\lambda-\frac{\mathrm{i}}{2}\right) &= \mathbb{I}+\frac{\mathrm{i}}{\lambda}\mathbb{P}, \qquad \mathbb{P} = \begin{pmatrix} J & -J^- \\ J^+ & -J\end{pmatrix}, \\ \hat{L}\left(\lambda+\frac{\mathrm{i}}{2}\right) &= \mathbb{I}+\frac{\mathrm{i}}{\lambda}\hat{\mathbb{P}}, \qquad \hat{\mathbb{P}} = \begin{pmatrix} J+1 & J^- \\ -J^+ & -J+1\end{pmatrix}. \end{aligned} \tag{3.13}$$

To obtain the charges in involution we expand it in powers of $\frac{1}{\lambda}$ as

$$\bar{\mathbb{K}}(\lambda) = \mathbb{I} + \frac{1}{\lambda}\bar{\mathbb{K}}^{(0)} + \frac{1}{\lambda^2}\bar{\mathbb{K}}^{(1)}, \tag{3.14}$$

where

$$\begin{aligned} \bar{\mathbb{K}}^{(0)} &= \mathrm{i}\begin{pmatrix} 2J+1 & 0 \\ 0 & -2J+1\end{pmatrix}, \\ \bar{\mathbb{K}}^{(1)} &= \begin{pmatrix} -J^2-\frac{1}{2}\{J^+,J^-\}-2J & -2JJ^- \\ -2JJ^+ & -J^2-\frac{1}{2}\{J^+,J^-\}+2J\end{pmatrix}. \end{aligned}$$

[2] In many of the algebraic manipulations, such as in that leading to the second equality below, we omit constant overall factors since from the relevant defining equations either they drop out or they produce an overall factor that is unimportant for our considerations.

Taking the trace we end up with

$$\bar{t}(\lambda) = \mathrm{tr}\{\bar{\mathbb{K}}(\lambda)\} = \mathbb{I} + \frac{\mathrm{i}}{\lambda} + \frac{\bar{t}^{(1)}}{\lambda^2}, \qquad \text{where} \quad \bar{t}^{(1)} \propto J^2 + \frac{1}{2}\{J^+, J^-\}. \tag{3.15}$$

Here $\bar{t}^{(1)}$ is the only non-trivial conserved quantity as is dictated by the commutation relations (2.11) because the expansion stops at $\frac{1}{\lambda^2}$. If we had to deal with higher terms in the expansion, we would have more conserved quantities (higher Casimir operators) as will be transparent in the subsequent sections, when dealing with higher rank algebras. This quantity will become the quadratic Casimir of E_2 after contraction.

Note that the conserved quantity is not the sl_2 Casimir operator (it is structurally an su_2-like Casimir), but rather an element of the Abelian part of the twisted Yangian. The reason is that the symmetry of the particular boundary model is not an sl_2 one, but simply $u(1)$ as dictated by the form of $\bar{\mathbb{K}}^{(0)}$. This is in accordance with [21, 22],

$$\left[\mathrm{tr}\{\bar{\mathbb{K}}(\lambda)\}, \bar{\mathbb{K}}^{(0)}_{ab}\right] = 0, \tag{3.16}$$

a relation that defines the exact symmetry of the transfer matrix. The transfer matrix that enjoys the full sl_2 symmetry will be presented subsequently.

Let us define $\bar{\mathbb{K}}^{(1)}_{12} = -2Y^-$ and $\bar{\mathbb{K}}^{(1)}_{21} = -2Y^+$. After performing the Inonü–Wigner contraction $J^\pm \to \frac{1}{\epsilon}P^\pm$, with $\epsilon \to 0$, we end up with

$$Y^\pm = JP^\pm, \tag{3.17}$$

precisely as in (1.2). Therefore we have the exchange relations (1.1) and seemingly it looks as if one can expand E_2 back to sl_2 (see [1]). However, this is not true, since J and $Y^\pm$ are elements of an extended algebra, i.e. the contracted sl_2 twisted Yangian $\mathcal{T}$ with exchange relations dictated by the quadratic equation (3.8). This will be more transparent in the following when constructing the N-tensor representation of the twisted Yangian.

3.1.1. The N-particle construction. In this section, we basically illustrate the presence of non-trivial co-products further manifesting the existence of the underlying non-trivial algebra that is the twisted Yangian of E_2. We define the N-tensor representation of the twisted Yangian as

$$\bar{\mathbb{T}}_0(\lambda) = L_{0N}\left(\lambda - \frac{\mathrm{i}}{2}\right)\cdots L_{01}\left(\lambda - \frac{\mathrm{i}}{2}\right)\hat{L}_{01}\left(\lambda + \frac{\mathrm{i}}{2}\right)\cdots \hat{L}_{0N}\left(\lambda + \frac{\mathrm{i}}{2}\right). \tag{3.18}$$

As before, an expansion in powers of $\frac{1}{\lambda}$ leads to

$$\bar{\mathbb{T}}(\lambda) = \mathbb{I} + \sum_{k=1}^{2N}\frac{\bar{\mathbb{T}}^{(k-1)}}{\lambda^k}, \tag{3.19}$$

with

$$\begin{aligned} \bar{\mathbb{T}}^{(0)}_0 &= \sum_{i=1}^{N}(\mathbb{P}_{0i} + \hat{\mathbb{P}}_{0i}), \\ \bar{\mathbb{T}}^{(1)}_0 &= -\left(\sum_{i>j}\mathbb{P}_{0i}\mathbb{P}_{0j} + \sum_{i<j}\hat{\mathbb{P}}_{0i}\hat{\mathbb{P}}_{0j} + \sum_{i,j}\mathbb{P}_{0i}\hat{\mathbb{P}}_{0j}\right) \end{aligned} \tag{3.20}$$

and so on. The first non-trivial conserved quantity in this case is obtained after taking the trace of $\bar{\mathbb{T}}^{(1)}$. One finds that

$$\bar{t}^{(1)} \propto \sum_{i=1}^{N}\left(J_i^2 + \frac{1}{2}\{J_i^+, J_i^-\}\right) + 4\sum_{i<j}J_iJ_j. \tag{3.21}$$

After performing the contraction, the first conserved quantity is given by

$$\bar{t}^{(1)} = \sum_{i=1}^{N} P_i^+ P_i^- . \tag{3.22}$$

Note that for $N = 1$ one simply obtains the expected E_2 Casimir emerging directly from (3.15) after contracting and keeping the highest order contribution. All quantities $\bar{t}^{(k)}$ (in the co-product form now) form, as dictated by the integrability condition (2.11), an Abelian algebra (family of commuting operators), which is part of the twisted Yangian of E_2. It is worth noting that after contraction we consistently keep only the highest order terms for each $\bar{t}^{(k)}$. This will be explained in more detail in section 4.

For the non-diagonal elements we have $\bar{\mathbb{T}}_{12} = -2\mathbb{Y}^-$ and $\bar{\mathbb{T}}_{21} = -2\mathbb{Y}^+$, where

$$\mathbb{Y}^{\pm} = \sum_{i=1}^{N} J_i P_i^{\pm} + 2 \sum_{i<j} J_i P_j^{\pm}, \tag{3.23}$$

with the subscript standing for the ith site in the N co-product sequence. Recall that the co-product for all generators of sl_2 is given by (1.4). On the other hand from (3.23), it is obvious that the two co-product of the underlying algebra, given by the two-particle case, is

$$\Delta(Y^{\pm}) = Y^{\pm} \otimes \mathbb{I} + \mathbb{I} \otimes Y^{\pm} + 2J \otimes P^{\pm}. \tag{3.24}$$

Comparison with (1.4) suggests that $Y^{(\pm)}$, J belong to a broader deformed algebra, which in the particular case is the contracted sl_2 twisted Yangian. Note the extra term appearing in the co-product, which suggests that we deal with a deformed algebra, that is the contracted sl_2 twisted Yangian, and not an extension to sl_2 as was claimed for instance in [1]. Study of representations shows inconsistencies, i.e. $\Delta(Y^{\pm})$, $\Delta(J)$ do not belong to $U(sl_2) \otimes U(sl_2)$ as one might have expected, but to a broader algebra the $\mathcal{T} \otimes \mathcal{A}$ ($\mathcal{T}$ is a co-ideal of $\mathcal{A}$, see also e.g. [21, 23–25]). Furthermore, these co-products do not satisfy (1.1), contrary to the naive co-products (1.5). If they were, this would have been surprising due to the fact that for one 'particle' the expansion stops at order $\frac{1}{\lambda^2}$ so the relevant exchange relations emanating from (3.8) are somehow truncated. However, once we consider the N 'particle' representation the expansion involves higher orders, and thus the associated exchange relations become more involved.

3.2. The reflection algebra and its contraction

We consider the representation $\mathbb{K}(\lambda)$ (2.8) of the reflection algebra with $\Theta = \frac{i}{2}$. The transfer matrix is given by (2.10). By choosing $K = \mathrm{diag}(-1, 1)$ and $K^+ = \mathrm{diag}(1, -1)$ we deal with a situation similar to the description above. We obtain the same conserved charges in involution, and also the associated transfer matrix enjoys the $u(1)$ symmetry. This is due the aforementioned equivalence of the twisted Yangian and reflection algebras for sl_2.

Consider now both $K = K^+ = \mathbb{I}$ in the transfer matrix t (2.10), that is we choose different boundary conditions for the associated physical system. We consider directly the N-particle Hamiltonian (N-tensor representation), (2.9) and (2.3) with $\theta_i = -\frac{\mathrm{i}}{2}$, that is we have

$$\mathbb{T}_0(\lambda) = L_{0N}\left(\lambda + \frac{\mathrm{i}}{2}\right) \cdots L_{01}\left(\lambda + \frac{\mathrm{i}}{2}\right) L_{01}\left(\lambda - \frac{\mathrm{i}}{2}\right) \cdots L_{0N}\left(\lambda - \frac{\mathrm{i}}{2}\right). \tag{3.25}$$

As before, an expansion in powers of $\frac{1}{\lambda}$ leads to

$$t^{(1)} \propto \sum_{i=1}^{N} \left(J_i^2 - \frac{1}{2}\{J_i^+, J_i^-\}\right) + 2 \sum_{i<j} \left(2J_i J_j - J_i^+ J_j^- - J_i^- J_j^+\right), \tag{3.26}$$

after we have taken the trace of the $1/\lambda^2$ term. Note the natural appearance of the quadratic Casimir operator for sl_2 for the one-particle case.

In this case after contraction we obtain

$$t^{(1)} \propto \sum_{i=1}^{N} P_i^+ P_i^- + 2 \sum_{i<j} P_i^+ P_j^-. \tag{3.27}$$

Also, we can safely say that

$$I = \sum_{i<j} P_i^- P_j^+ \tag{3.28}$$

is also a conserved quantity because $P_i^- P_i^+$ is the Casimir associated at each site and commutes with all elements acting on the same site. It is worth noting that although both conserved quantities coincide in the one 'particle' situation, they are obviously different when more 'particles' are involved, $\bar{t}^{(1)} \neq t^{(1)}$, that is their co-products are different. In this case, the transfer matrix enjoys the full sl_2 symmetry and it is natural that we obtain the sl_2 Casimir as a conserved quantity from the transfer matrix expansion.

4. The E_2^c extended algebra

In this section, we aim at constructing the centrally extended E_2^c algebra. To achieve this, we start from the gl_3 spin chain and break the symmetry down to $sl_2 \otimes u(1)$ by implementing appropriate boundary conditions. It has been known [26] that by implementing appropriate boundary conditions one can break the gl_n symmetry of a spin chain model to $gl_l \otimes gl_{n-l}$, where l is an integer depending on the choice of boundary. We shall exploit this phenomenon in order to perform a contraction of the boundary algebra to E_2^c.

The gl_n algebra is

$$[J_{ij}, J_{kl}] = \delta_{il} J_{kj} - \delta_{jk} J_{il}, \qquad i = 1, 2, \ldots, n. \tag{4.1}$$

It is generated by

$$J^{+(i)} = J^{i+1i}, \qquad J^{-(i)} = J^{ii+1}, \qquad e^{(i)} = J^{ii}. \tag{4.2}$$

Define $s^{(k)} = e^{(k)} - e^{(k+1)}$, then the following commutation relations are valid:

$$[J^{+(k)}, J^{-(l)}] = \delta_{kl} s^{(k)}, \qquad [s^{(k)}, J^{\pm(l)}] = \pm(2\delta_{kl} - \delta_{kl+1} - \delta_{kl-1}) J^{\pm(l)}, \tag{4.3}$$

and $\sum_{i=1}^n e^{(i)}$ belongs to the center of the algebra.

From now on, we focus on the case of gl_3. The L matrix is expressed as in (2.12), where $\mathbb{P}$ in terms of the gl_3 elements takes the form

$$\mathbb{P} = \begin{pmatrix} e^{(1)} & J^{-(1)} & \Lambda^+ \\ J^{+(1)} & e^{(2)} & J^{-(2)} \\ \Lambda^- & J^{+(2)} & e^{(3)} \end{pmatrix}, \qquad \text{where} \qquad \Lambda^\pm = \pm[J^{\pm(1)}, J^{\pm(2)}]. \tag{4.4}$$

Consider next the N-tensor representation of the reflection algebra (2.9), (2.3), with $\theta_i = 0$ and L given in (2.12) and (4.4). We choose as K the following diagonal matrix (for a more general solution see [26, 27])

$$K(\lambda) = k = \mathrm{diag}(1, 1, -1) \tag{4.5}$$

and expand $\mathbb{T}(\lambda)$, given by

$$\mathbb{T}_0(\lambda) = L_{0N}(\lambda) \cdots L_{01}(\lambda) k L_{01}^{-1}(-\lambda) \cdots L_{0N}^{-1}(-\lambda), \tag{4.6}$$

as (see also [21])

$$\mathbb{T}(\lambda) = k + \sum_{k=1}^{2N} \frac{\mathbb{T}^{(k-1)}}{\lambda^k}, \tag{4.7}$$

where

$$\begin{aligned} \mathbb{T}^{(0)} &= \mathrm{i}\sum_{i=1}^{N} (\mathbb{P}_{0i} k + k\mathbb{P}_{0i}), \\ \mathbb{T}^{(1)} &= -\sum_{i>j} \mathbb{P}_{0i}\mathbb{P}_{0j} k - k\sum_{i<j} \mathbb{P}_{0i}\mathbb{P}_{0j} - \sum_{i,j=1}^{N} \mathbb{P}_{0i} k \mathbb{P}_{0j} - k\sum_{i=1}^{N} \mathbb{P}_{0i}^2, \end{aligned} \tag{4.8}$$

for the first two terms. Before we proceed, let us first recall what happens when $k = \mathbb{I}$. In this case, the open transfer matrix enjoys the full gl_3 symmetry (see e.g. [21, 22, 26]) and from the trace of $\mathbb{T}^{(1)}$ we obtain the quadratic Casimir. For instance, for one-'particle' ($N = 1$) we obtain

$$\begin{aligned} C = \mathrm{tr}\{\mathbb{P}^2\} &= (e^{(1)})^2 + (e^{(2)})^2 + (e^{(3)})^2 + J^{-(1)}J^{+(1)} + J^{-(2)}J^{+(2)} + \Lambda^-\Lambda^+ \\ &\quad + J^{+(1)}J^{-(1)} + J^{+(2)}J^{-(2)} + \Lambda^+\Lambda^-. \end{aligned} \tag{4.9}$$

Let us now come back to the situation where k is given by (4.5). Then

$$\mathbb{T}^{(0)} = 2\mathrm{i}\sum_{j=1}^{N} \begin{pmatrix} e_j^{(1)} & J_j^{-(1)} & 0 \\ J_j^{+(1)} & e_j^{(2)} & 0 \\ 0 & 0 & -e_j^{(3)} \end{pmatrix}. \tag{4.10}$$

Clearly, what remains consists of the $sl_2 \otimes u(1)$ algebra. Specifically, $\left(\sum_j J^{\pm(1)}, \sum_j s_j^{(1)}\right)$ satisfies the sl_2 commutation relations, whereas $\sum_i e_i^{(3)}$ commutes with everything. The first two conserved quantities are given by taking the trace over the auxiliary space in $\mathbb{T}^{(0)}, \mathbb{T}^{(1)}$. We obtain

$$t^{(0)} \propto \sum_{j=1}^{N} \left(c_j - 2e_j^{(3)}\right), \qquad \text{where} \quad c_j = e_j^{(1)} + e_j^{(2)} + e_j^{(3)} \tag{4.11}$$

and

$$\begin{aligned} t^{(1)} \propto \sum_{j=1}^{N} &\left(\left(e_j^{(1)}\right)^2 + \left(e_j^{(2)}\right)^2 - \left(e_j^{(3)}\right)^2 + J_i^{-(1)}J_i^{+(1)} + J_i^{+(1)}J_i^{-(1)} - c_j + 3e_j^{(3)}\right) \\ &+ 2\sum_{i<j} \left(e_i^{(1)}e_j^{(1)} + e_i^{(2)}e_j^{(2)} - e_i^{(3)}e_j^{(3)} + J_i^{-(1)}J_j^{+(1)} + J_i^{+(1)}J_j^{-(1)}\right). \end{aligned} \tag{4.12}$$

The one 'particle' Hamiltonian $t^{(1)}$ is then

$$t^{(1)} \propto (e^{(1)})^2 + (e^{(2)})^2 - (e^{(3)})^2 + J^{-(1)}J^{+(1)} + J^{+(1)}J^{-(1)}, \tag{4.13}$$

where we have omitted the linear in the generators terms corresponding to the first line of (4.12) since they commute with everything. For notational convenience, we set

$$s^{(1)} \equiv 2J, \qquad e^{(3)} \equiv 2\tilde{J}, \qquad J^{-(1)} \equiv -J^-, \qquad J^{+(1)} \equiv J^+. \tag{4.14}$$

Then one finds

$$t^{(1)} \propto J^2 - \tfrac{1}{2}\{J^+, J^-\} - \tilde{J}^2 - c\tilde{J}. \tag{4.15}$$

From the first integral of motion it is clear that $e^{(3)} = 2\tilde{J}$ is also a conserved quantity, so the second charge may be written as

$$I^{(0)} = \tilde{c} - 2\tilde{J}, \qquad I^{(1)} = J^2 - \tfrac{1}{2}\{J^+, J^-\} - \tilde{J}^2 \qquad \tilde{c} = e^{(1)} + e^{(2)} \tag{4.16}$$

$\tilde{c}$ is obviously a sl_2 central element.

Similarly, the N-tensor representations are given, respectively, by

$$\begin{aligned}\mathbb{I}^{(0)} &= \sum_{j=1}^{N} (\tilde{c}_j - 2\tilde{J}_j),\\ \mathbb{I}^{(1)} &= \sum_{j=1}^{N}\left(J_j^2 - \frac{1}{2}\{J_j^+, J_j^-\} - \tilde{J}_j^2\right) + 4\sum_{i<j}\left(J_i J_j - \frac{1}{2}(J_i^+ J_j^- + J_i^- J_j^+) - \tilde{J}_i\tilde{J}_j\right).\end{aligned} \tag{4.17}$$

We shall exploit the breaking of the symmetry due to the presence of non-trivial integrable boundaries to obtain the centrally extended E_2^c algebra. Consider the following contraction[3]

$$J^{\pm} = \frac{1}{\sqrt{2\epsilon}} P^{\pm}, \qquad J = \frac{1}{2}\left(T + \frac{F}{\epsilon}\right), \qquad \tilde{J} = -\frac{F}{2\epsilon}, \qquad \epsilon \to 0. \tag{4.18}$$

Then one obtains the following commutation relations that define the E_2^c algebra,

$$[P^+, P^-] = -2F, \qquad [T, P^{\pm}] = \pm P^{\pm}, \tag{4.19}$$

where F is an *exact* central element of the algebra. It is obvious that the conserved quantities, for the one-site case, after contracting and keeping the leading order contribution are

$$I^{(0)} = F, \qquad I^{(1)} = TF - \tfrac{1}{2}\{P^+, P^-\}. \tag{4.20}$$

The N-site representation follows immediately from (4.17)

$$\mathbb{I}^{(1)} \sim \sum_{i=1}^{N}\left(F_i T_i - \frac{1}{2}\{P_i^+, P_i^-\}\right) + 2\sum_{i<j}\left(F_i T_j + T_i F_j - \{P_i^+, P_j^-\}\right). \tag{4.21}$$

Let us now focus on the $N = 1$ and see more precisely how one obtains higher Casimir operators of E_2^c from the expansion of the transfer matrix $t(\lambda) = \sum_{k=1}^{2N} \frac{t^{(k-1)}}{\lambda^k}$. Recall the $N = 1$ representation of the reflection algebra

$$\begin{aligned}\mathbb{T}(\lambda) = L(\lambda)k\hat{L}(\lambda) &= \left(1 + \frac{\mathrm{i}}{\lambda}\mathbb{P}\right)k\left(1 + \frac{\mathrm{i}}{\lambda}\mathbb{P} - \frac{1}{\lambda^2}\mathbb{P}^2 - \frac{\mathrm{i}}{\lambda^3}\mathbb{P}^3 + \frac{1}{\lambda^4}\mathbb{P}^4 \cdots\right)\\ &= k + \frac{\mathrm{i}}{\lambda}(\mathbb{P}k + k\mathbb{P}) - \frac{1}{\lambda^2}(\mathbb{P}k\mathbb{P} + k\mathbb{P}^2) - \frac{1}{\lambda^2}(\mathbb{P}k\mathbb{P}^2 + k\mathbb{P}^3)\cdots,\end{aligned} \tag{4.22}$$

where k is as given in (4.5) then

$$t^{(k-1)} \propto \sum_{a,b}\left(\mathbb{P}_{ab}k_{bb}\mathbb{P}_{ba}^{k-1} + k_{aa}\mathbb{P}_{aa}^{k}\right). \tag{4.23}$$

Before contraction $t^{(k)}$ are the higher Casimir operators of $sl_2 \otimes u_1$, after contracting one has to consistently keep only the highest order contribution in the $\frac{1}{\epsilon}$ expansion of each $t^{(k)}$. Then each one of $t^{(k)}$ commutes by construction with E_2^c. We have already explicitly computed the quadratic one (4.20), and the derivation of higher Casimir $t^{(k)}$ is then simply a matter

[3] This contraction is known in the mathematics literature as a Saletan contraction and is distinct from the Inonü–Winger contraction. It is the analog of the so-called Penrose limit in gravity that constructs a plane wave starting from any gravitational background by magnifying the region around a null geodesic and it was first used in the string literature in WZW models [28]. More recently, the Penrose limit has been taken in various supersymmetric brane solutions of string and M-theory [29] and has been instrumental in understanding issues within the AdS/CFT correspondence involving sectors of large quantum numbers [30].

of involved algebraic computations, since the generic form is known (4.23). It is clear that since each one of $t^{(k)}$ commutes with E_2^c the transfer matrix also commutes. This logic may be generalized for performing contraction to any higher rank algebra or q-deformed algebra (see the following section), the same applies for generic N. In fact, this argument holds independently of the context one realizes the contraction (see e.g. [28]). More precisely, having in general a set of Casimir operators of say the gl_n algebra after contraction one consistently should keep the highest order contribution in order to obtain the contracted Casimir quantities. Depending on the rank of the considered algebra the expansion of $t(\lambda)$ should truncate at some point—note that expressions (4.22) and (4.23) are generic and hold for any gl_n—or in other words, the higher Casimir quantities should be trivial combinations of the lower ones. This is a quite intricate technical point; however, this is beyond the scope of this paper.

We may obtain a construction similar to that above by re-parametrizing the L operator—the solution of the fundamental equation (2.2), with R being $\mathcal{Y}(sl_2)$. Define below the L, $\hat{L}$ operators

$$L(\lambda) = 1 + \frac{\mathrm{i}}{\lambda}\mathbb{P}, \qquad \hat{L}(\lambda) = 1 + \frac{\mathrm{i}}{\lambda}\hat{\mathbb{P}}, \tag{4.24}$$

where

$$\mathbb{P} = \begin{pmatrix} \tilde{J} + J & -J^- \\ J^+ & \tilde{J} - J \end{pmatrix}, \qquad \hat{\mathbb{P}} = \begin{pmatrix} -\tilde{J} + J + 1 & -J^- \\ J^+ & -\tilde{J} - J + 1 \end{pmatrix}. \tag{4.25}$$

To obtain a Casimir-like quantity, it is more natural to consider the open spin chain system

$$\mathbb{T}_0(\lambda) = L_{0N}(\lambda)\cdots L_{01}(\lambda)\hat{L}_{01}(\lambda)\cdots\hat{L}_{0N}(\lambda). \tag{4.26}$$

The charges in involution may again be obtained via an appropriate expansion. We omit here the relevant details for brevity. Finally, after expanding and contracting we obtain from the $1/\lambda^2$ term

$$\mathbb{I}^{(1)} = \sum_{j=1}^{N}\left(J_j^2 - \frac{1}{2}\{J_j^+, J_j^-\} - \tilde{J}_j^2\right) + 4\sum_{i<j}\left(J_i J_j - \frac{1}{2}\left(J_i^+ J_j^- + J_i^- J_j^+\right)\right). \tag{4.27}$$

After the contraction, we obtain for the first non-trivial one-particle charge

$$I^{(1)} = TF - \tfrac{1}{2}\{P^+, P^-\}, \tag{4.28}$$

whereas the N-particle charge becomes

$$\mathbb{I}^{(1)} \sim \sum_{i<j} F_i F_j. \tag{4.29}$$

Note that the one 'particle' conserved quantities (4.20) and (4.28) coincide, whereas the N 'particle' charges (4.21) and (4.29) are different. In fact, the underlying symmetries in the two descriptions are different. In the second case, the remaining symmetry after implementing the boundary is sl_2 (the $u(1)$ symmetry is hidden) whereas in the first description it is $sl_2 \otimes u(1)$. The description of (4.17) gives the expected co-product associated with the Casimir of $sl_2 \otimes u(1)$, so that in this sense it is a more natural description.

5. The $U_q(E_2^c)$ algebra from $U_q(sl_2) \otimes u(1)$

We shall now turn and study in our context the q-deformed situation. More precisely, we shall focus on the construction of the centrally extended $U_q(E_2^c)$ algebra from $U_q(sl_2)\otimes u(1)$. It was shown in [31] that applying special boundary conditions in an open $U_q(gl_n)$ spin chain (in the fundamental representation) breaks the symmetry to $U_q(gl_l) \otimes U_q(gl_{n-l})$, where l is an integer associated with the choice of boundary. This statement was generalized for generic algebraic objects, independently of the choice of representation (about boundary quantum algebras and for generic boundary conditions see [25, 32]). The case with no central extension, that is $U_q(E_2)$, has been obtained in earlier works [33] from the $U_q(sl_2)$ algebra via a Inonü–Wigner-type contraction.

Before we proceed, let us first review some relevant facts regarding the $U_q(gl_n)$ algebra [9–12]. Let

$$a_{ij} = 2\delta_{ij} - (\delta_{i\ j+1} + \delta_{i\ j-1}), \qquad i, j = 1, 2, \ldots, n-1 \tag{5.1}$$

be the Cartan matrix of the Lie algebra sl_n. The quantum enveloping algebra $U_q(sl_n)$ with the Chevalley–Serre generators [9, 11]

$$e_i, \quad f_i, \quad q^{\pm\frac{s_i}{2}}, \qquad i = 1, 2, \ldots, n-1 \tag{5.2}$$

obeys the defining relations

$$\begin{aligned} &\left[q^{\pm\frac{s_i}{2}}, q^{\pm\frac{s_j}{2}}\right] = 0 \qquad q^{\frac{s_i}{2}}\, e_j = q^{\frac{1}{2}a_{ij}} e_j\, q^{\frac{s_i}{2}} \qquad q^{\frac{s_i}{2}}\, f_j = q^{-\frac{1}{2}a_{ij}} f_j\, q^{\frac{s_i}{2}}, \\ &[e_i, f_j] = \delta_{ij} \frac{q^{s_i} - q^{-s_i}}{q - q^{-1}}, \qquad i, j = 1, 2, \ldots, n-1. \end{aligned} \tag{5.3}$$

They also satisfy the q-deformed Serre relations, omitted here for brevity (see e.g. [9]). The above generators form the $U_q(sl_n)$ algebra and also, $q^{\pm s_i} = q^{\pm(\epsilon_i - \epsilon_{i+1})}$. The $U_q(gl_n)$ algebra is derived by adding to $U_q(sl_n)$ the elements $q^{\pm\epsilon_i}$ $i = 1, \ldots, n$ so that $q^{\sum_{i=1}^n \epsilon_i}$ belongs to the center (for more details see [9]).[4]

The algebra $U_q(gl_n)$ is equipped with a co-product $\Delta : U_q(gl_n) \to U_q(gl_n) \otimes U_q(gl_n)$, acting as

$$\Delta(y) = q^{-\frac{s_i}{2}} \otimes y + y \otimes q^{\frac{s_i}{2}}, \qquad y \in \{e_i, f_i\}, \qquad \Delta(q^{\pm\frac{\epsilon_i}{2}}) = q^{\pm\frac{\epsilon_i}{2}} \otimes q^{\pm\frac{\epsilon_i}{2}}. \tag{5.4}$$

We shall focus here on the (trigonometric) $\mathbb{U}_q(\widehat{gl_n})$ R-matrix which is given by [9]

$$R(\lambda) = a(\lambda) \sum_{i=1}^n \hat{e}_{ii} \otimes \hat{e}_{ii} + b(\lambda) \sum_{i \neq j=1}^n \hat{e}_{ii} \otimes \hat{e}_{jj} + c \sum_{i \neq j=1}^n \mathrm{e}^{-sgn(i-j)\lambda} \hat{e}_{ij} \otimes \hat{e}_{ji}, \tag{5.5}$$

where $R \in \mathrm{End}((\mathbb{C}^n)^{\otimes 2})$ and $\hat{e}_{ij}$ are $n \times n$ matrices with elements $(\hat{e}_{ij})_{kl} = \delta_{ik}\delta_{jl}$. We also define for notational convenience

$$a(\lambda) = \sinh \mu(\lambda + \mathrm{i}), \qquad b(\lambda) = \sinh \mu\lambda, \qquad c = \sinh \mathrm{i}\mu, \qquad q = e^{i\mu}. \tag{5.6}$$

The associated L operator in this case may be written in the following form [34],

$$L(\lambda) = \mathrm{e}^{\mu\lambda} L^+ - \mathrm{e}^{-\mu\lambda} L^-, \tag{5.7}$$

where

$$L^+ = \sum_{i \leqslant j} \hat{e}_{ij} \otimes t_{ij}, \qquad L^- = \sum_{i \geqslant j} \hat{e}_{ij} \otimes t^-_{ij}. \tag{5.8}$$

For definitions of the elements t_{ij} and t^-_{ij} we refer to the appendix.

As in the previous sections we are mostly interested in representations of the reflection algebra (2.5), given in (2.9) and (2.3), where now L is given by (5.7). We also choose the homogeneity parameter $\Theta = 0$. To define $L^{-1}(-\lambda)$ is quite intricate and in general is

[4] We have changed the notation we have used for the generators of the undeformed gl_n case in order to conform with the literature we refer to. The correspondence with them is $e_i \to J^{+(i)}$, $f_i \to J^{-(i)}$, $\epsilon_i \to e^{(i)}$ and $s_i \to s^{(i)}$. In the limit $q \to 1$, we recover from (5.3) the commutators (4.3).

expressed in powers of $e^{-\mu\lambda}$. For our purposes, here we are interested in the highest order of the expansion (as $\lambda \to \infty$) [25]

$$L^{-1}(-\lambda) \sim \hat{L}^{+} + \mathcal{O}(e^{-2\mu\lambda}), \qquad \text{where} \quad \hat{L}^{+} = \sum_{i\geqslant j} \hat{e}_{ij} \otimes \hat{t}_{ij}. \tag{5.9}$$

For the definition of $\hat{t}_{ij}$ we refer again to the appendix as well. One can readily check that $L^{-}\hat{L}^{+} = \mathbb{I}$ (with no proportionality factor). The open transfer matrix is defined in (2.10), and for simplicity we shall set henceforth $K^{+} \propto M$, where M is a matrix that in the $U_q(gl_n)$ series—in the homogeneous gradation (for details see e.g. [31, 25])—has the form

$$M = \sum_{j} q^{n-2j+1} \hat{e}_{jj} \qquad \text{and} \qquad [M_1 M_2, R_{12}(\lambda)] = 0. \tag{5.10}$$

Consider a c-number solution of the reflection equation of type (5.17), then its asymptotic behavior as $\lambda \to \infty$ is $K(\lambda \to \infty) \sim K^{(0)} + \mathcal{O}(e^{-2\mu\lambda})$. The representation of the reflection algebra for one 'particle' as $\lambda \to \infty$ becomes ($T \to L$ and $\mathbb{T} \to \mathbb{K}$)

$$\mathbb{K}(\lambda \to \infty) = L^{+} K^{(0)} \hat{L}^{+}. \tag{5.11}$$

We will only consider diagonal c-number solutions of the diagonal boundary conditions that break $U_q(gl_n)$ to $U_q(gl_l) \otimes U_q(gl_{n-l})$, as was first shown in [31]. If both K^{+}, $K \propto \mathbb{I}$ then the open transfer matrix enjoys the full $U_q(gl_n)$ symmetry (see e.g. [25, 31, 32] and references therein). In this case, of trivial boundary conditions that preserve the full symmetry one obtains the Casimir of the associated $U_q(gl_n)$ algebra. Indeed as $\lambda \to \infty$ we obtain[5]

$$t^{+} = \operatorname{tr}(M\, L^{+}\hat{L}^{+}) = \sum_{i\leqslant j} q^{n-2i+1} t_{ij} \hat{t}_{ji}. \tag{5.14}$$

This is perhaps the most natural and simplest way to obtain Casimir operators of q-deformed algebras. For instance, in the case of $U_q(gl_3)$, we obtain

$$L^{+} = \begin{pmatrix} q^{\epsilon_1} & t_{12} & t_{13} \\ 0 & q^{\epsilon_1} & t_{23} \\ 0 & 0 & q^{\epsilon_1} \end{pmatrix}, \qquad \hat{L}^{+} = \begin{pmatrix} q^{\epsilon_1} & 0 & 0 \\ \hat{t}_{21} & q^{\epsilon_1} & 0 \\ \hat{t}_{31} & \hat{t}_{23} & q^{\epsilon_1} \end{pmatrix}. \tag{5.15}$$

Then, the explicit expression of the Casimir operator is given by

$$\begin{aligned} t^{+} = q^2 q^{2\epsilon_1} + q^{2\epsilon_2} + q^{-2} q^{2\epsilon_3} + (q - q^{-1})^2 \left(q^{\epsilon_1+\epsilon_2+1} f_1 e_1 + q^{\epsilon_2+\epsilon_3-1} f_2 e_2\right) \\ + (q - q^{-1})^2 q^{\epsilon_1+\epsilon_3} (q f_2 f_1 - f_1 f_2)(e_1 e_2 - q^{-1} e_2 e_1). \end{aligned} \tag{5.16}$$

The above expression is quite compact and depends in a straightforward manner on the algebra generators (see also [35]). Note that by construction the quantity t^{+} belongs to the center of $U_q(gl_3)$. Recall that in this case the transfer matrix enjoys the full $U_q(gl_3)$ symmetry [25, 31], thus all the charges in involution generated from the transfer matrix belong to the center of $U_q(gl_3)$. Note that, using (5.16) and (5.13), the sum of t^{+} and t^{-} reduces to the gl_3 quadratic Casimir (4.9) in the isotropic limit $q \to 1$. In general, the spectrum of these Casimir-type quantities associated with any $U_q(gl_n)$ invariant open spin system may be readily computed

[5] Note that as $\lambda \to -\infty$ one obtains another Casimir operator for the deformed algebra

$$t^{-} = \operatorname{tr}(M\, L^{-}\hat{L}^{-}) = \sum_{i\geqslant j} q^{n-2i+1} t^{-}_{ij} \hat{t}^{-}_{ji}. \tag{5.12}$$

In the $U_q(gl_3)$ case in particular it reduces to

$$\begin{aligned} t^{-} = q^2 q^{-2\epsilon_1} + q^{-2\epsilon_2} + q^{-2} q^{-2\epsilon_3} + (q - q^{-1})^2 (q^{-\epsilon_1-\epsilon_2+1} e_1 f_1 + q^{-\epsilon_2-\epsilon_3-1} e_2 f_2) \\ + (q - q^{-1})^2 q^{-\epsilon_1-\epsilon_3} (q^{-1} e_1 e_2 - e_2 e_1)(f_2 f_1 - q f_1 f_2). \end{aligned} \tag{5.13}$$

via Bethe ansatz techniques for any representation. The spectrum and Bethe ansatz equations are known for these open spin chains [36]. Hence, expanding it in powers of $\mathrm{e}^{-\lambda}$ will provide the eigenvalues of the associated Casimir operators.

We are basically interested in obtaining $U_q\left(E_2^c\right)$ as a contraction of $U_q(sl_2) \otimes u(1)$, which again is a boundary symmetry. We shall focus henceforth on $U_q(gl_3)$ and on diagonal K-matrices of the form [27]

$$K(\lambda) = \mathrm{diag}(\mathrm{e}^{\mu\lambda}, \mathrm{e}^{\mu\lambda}, \ -\mathrm{e}^{-\mu\lambda}). \tag{5.17}$$

We take into account the asymptotics L^+, $\hat{L}^+$, in the $U_q(gl_3)$ (5.15) and $K^{(0)} = \mathrm{diag}(1, 1, 0)$. We may now explicitly write

$$\mathbb{K}(\lambda \to \infty) \sim \mathbb{K}^+ = \begin{pmatrix} q^{2\epsilon_1} + t_{12}\hat{t}_{21} & t_{12}q^{\epsilon_2} & 0 \\ q^{\epsilon_2}\hat{t}_{21} & q^{2\epsilon_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.18}$$

where using the appendix we have

$$t_{ii} = q^{\epsilon_i}, \qquad t_{12} = (q - q^{-1})q^{-1/2}q^{\frac{\epsilon_1+\epsilon_2}{2}} f_1, \qquad \hat{t}_{21} \equiv (q - q^{-1})q^{-1/2}q^{\frac{\epsilon_1+\epsilon_2}{2}} e_1. \tag{5.19}$$

It is also convenient to implement the following identifications:

$$e_1 \equiv J^+, \qquad f_1 \equiv -J^-, \qquad \epsilon_1 - \epsilon_2 \equiv 2J, \qquad \epsilon_1 + \epsilon_2 = -2\tilde{J}, \tag{5.20}$$

clearly $\tilde{J}$ is the central element of $U_q(sl_2)$. The asymptotic expression of the transfer matrix as $\lambda \to \infty$ provides the first conserved quantity (higher terms in the expansion give rise to higher charges)

$$t^+ = \mathrm{tr}\{M\mathbb{K}^+\}, \tag{5.21}$$

where from (5.10) the matrix $M = \mathrm{diag}(q^2, 1, q^{-2})$. We conclude that

$$t^+ \propto q^{-2\tilde{J}}(q^{2J+1} + q^{-2J-1} - (q - q^{-1})^2 J^- J^+), \tag{5.22}$$

which is the Casimir operator of $U_q(sl_2)$ and $q^{-\tilde{J}}$ is apparently central element of $U_q(sl_2)$. This is somehow expected given that the associated transfer matrix enjoys the exact $U_q(sl_2) \times u(1)$ symmetry (see, e.g. [25, 31, 32] for details on the proof). The $u(1)$ charge is obtained from $\lambda \to -\infty$ asymptotic behavior of the transfer matrix and it is (we omit the technical details for brevity)

$$t^- = \epsilon_3. \tag{5.23}$$

In the N-tensor representation one obtains as expected the following non-local quantities

$$\begin{aligned} &t^{+(N)} \propto \Delta^{(N)}(q^{-2\tilde{J}})(q\Delta^{(N)}(q^{2J}) + q^{-1}\Delta^{(N)}(q^{-2J}) - (q - q^{-1})^2\Delta^{(N)}(J^-)\Delta^{(N)}(J^+)), \\ &t^{-(N)} = \Delta^{(N)}(\epsilon_3), \end{aligned} \tag{5.24}$$

where the indicated N co-products are derived via (5.4) by iteration. Higher order Casimir operators of the q-deformed algebra are obtained by considering the expansion of the open transfer matric in powers of $\mathrm{e}^{\pm 2\mu\lambda}$; the same arguments—for the contracted version as well—hold as in the rational case described in the previous sections.

Consider the Saletan-type contraction (4.18) after also setting the deformation parameter to $q = \mathrm{e}^{\epsilon\eta}$. This will lead to the deformed centrally extended $U_q\left(E_2^c\right)$ algebra. We obtain

$$[T, P^{\pm}] = \pm 2P^{\pm}, \qquad [P^+, P^-] = -2F_\eta, \qquad F_\eta = \frac{\sinh(\eta F)}{\eta}, \tag{5.25}$$

where F_η is an *exact* central element of the algebra. The associated co-products emanating from (5.4) are given by

$$\begin{aligned}
&\Delta(P^{\pm}) = \mathrm{e}^{-\frac{\eta F}{2}} \otimes P^{\pm} + P^{\pm} \otimes q^{\frac{\eta F}{2}},\\
&\Delta(\mathrm{e}^{\pm\eta F}) = \mathrm{e}^{\pm\eta F} \otimes \mathrm{e}^{\pm\eta F},\\
&\Delta(T) = \mathbb{I} \otimes T + T \otimes \mathbb{I}.
\end{aligned} \tag{5.26}$$

Note that, although the algebra (5.25) is an E_2^c one for the central extension F_η, what appears in the co-products is F itself.

The associated Casimir follows from (5.22) after we have performed the contraction. We find that

$$C = \mathrm{e}^{\eta F}\big(2\cosh(\eta F) + 2\eta^2\epsilon\big(T F_\eta - \tfrac{1}{2}\{P^+, P^-\}\big)\big). \tag{5.27}$$

It is straightforward to check that the latter quantity commutes with all the elements of the algebra $U_q\big(E_2^c\big)$. The N-co-product Casimir follows from (5.26) by iteration. Given that $F$is a central element of the constructed algebra we conclude that

$$I_\eta = T F_\eta - \tfrac{1}{2}\{P^+, P^-\} \tag{5.28}$$

is also a conserved quantity. It is clear that in the isotropic limit $\eta \to 0$ the algebra reduces to (4.19) and the associated Casimir operator to (4.28).

6. Discussion

The main theme of this paper is that by exploiting the symmetry breaking mechanism due to the presence of integrable boundary conditions, one can naturally construct contracted and centrally extended algebras.

In order to simplify the analysis and clearly demonstrate the main ideas, we restricted our discussion to examples involving originally sl_2 or gl_3 symmetry. A natural extension of the present work is to consider generic symmetry breaking of type G $\otimes$ H where G and H are generic algebras (H $\subset$ G), and then follow a contraction procedure similar to those for ordinary Lie and current algebras (see e.g. [28]). Also, note that the generic study of boundary symmetry breaking for higher rank algebras is presented in [25, 26, 31], and the super-symmetric case is a work in progress at the moment.

We were able to obtain the twisted Yangian of E_2 as well as the centrally extended E_2^c and $U_q\big(E_2^c\big)$ algebras via suitable contractions of gl_3 and $U_q(gl_3)$ algebras. Naturally, one may wonder if one could have started the whole analysis by directly considering the E_2, E_2^c and $U_q\big(E_2^c\big)$ algebras. However, such an approach would require knowledge of the universal R-matrices associated with E_2, E_2^c and $U_q\big(E_2^c\big)$. In general, such a derivation is an intricate issue, so the approach we followed here is admittedly the most straightforward and simplest one. Nonetheless it is clear according to the original works in [7, 9, 11] that linear exchange relations between the universal R-matrix and co-products of the charges of the Yangian or the affine q-algebra (see expressions of non-local charges in (3.20), (3.23), (3.24), (4.8), (4.10)) provide the exact form of the R-matrix. So our results may in addition be utilized for exactly deriving the universal R-matrix associated with the Yangian of E_2, E_2^c or the q-deformed E_2^c.

A more ambitious task is to explore and possibly apply our methods in the context of the AdS/CFT correspondence, where certain quantum group structures and supersymmetric centrally extended algebras arise (see, for instance, [37–39]). The pertinent question is whether one can use the generic context described here in order to uncover the full underlying algebraic structure, that is to extract by the methodology proposed the associated centrally extended algebra. That would be very important in understanding fundamental aspects of the AdS/CFT correspondence finitely beyond the perturbative level.

We hope to address the aforementioned significant issues in future works.

Appendix

In this appendix, we shall introduce some useful quantities (elements of $U_q(gl_n)$). Define $\mathcal{E}_{i\ i+1} = \hat{\mathcal{E}}_{i\ i+1} = e_i$ and $\mathcal{E}_{i+1\ i} = \hat{\mathcal{E}}_{i+1\ i} = f_i$, $i = 1, 2, \ldots, n-1$ and for $|i-j| > 1$, whereas

$$\begin{aligned}
\mathcal{E}_{ij} &= \frac{1}{|i-j|-1} \sum_{k=\min(i,j)+1}^{\max(i,j)-1} (\mathcal{E}_{ik}\, \mathcal{E}_{kj} - q^{\mp 1} \mathcal{E}_{kj}\, \mathcal{E}_{ik}), \qquad j \lessgtr k \lessgtr i, \\
\hat{\mathcal{E}}_{ij} &= \frac{1}{|i-j|-1} \sum_{k=\min(i,j)+1}^{\max(i,j)-1} (\hat{\mathcal{E}}_{ik} \hat{\mathcal{E}}_{kj} - q^{\pm 1} \hat{\mathcal{E}}_{kj} \hat{\mathcal{E}}_{ik}), \qquad j \lessgtr k \lessgtr i, \\
& i, j = 1, 2, \ldots, n.
\end{aligned} \tag{A.1}$$

Also define

$$\begin{aligned}
& t_{ij} = (q-q^{-1}) q^{-\frac{1}{2}} q^{\frac{\epsilon_i}{2}} q^{\frac{\epsilon_j}{2}}\, \mathcal{E}_{ji}, \quad i<j, \quad t^-_{ij} = -(q-q^{-1}) q^{\frac{1}{2}} q^{-\frac{\epsilon_i}{2}} q^{-\frac{\epsilon_j}{2}}\, \mathcal{E}_{ji}, \quad i>j \\
& \hat{t}_{ij} = (q-q^{-1}) q^{-\frac{1}{2}} q^{\frac{\epsilon_i}{2}} q^{\frac{\epsilon_j}{2}}\, \hat{\mathcal{E}}_{ji}, \quad i>j, \quad \hat{t}^-_{ij} = -(q-q^{-1}) q^{\frac{1}{2}} q^{-\frac{\epsilon_i}{2}} q^{-\frac{\epsilon_j}{2}}\, \hat{\mathcal{E}}_{ji}, \quad i<j, \\
& t_{ii} = \hat{t}_{ii} = (t^-_{ii})^{-1} = (\hat{t}^-_{ii})^{-1} = q^{\epsilon_i}.
\end{aligned} \tag{A.2}$$

References

[1] Gilmore R 1974 *Lie Groups, Lie Algebras, and Some of Their Applications* (New York: Wiley)
[2] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375
[3] Olshanski G I 1996 Twisted Yangians and infinite-dimensional classical Lie algebras *Quantum Groups (Lecture Notes in Mathematics* vol 1510*)* ed P P Kulish (Berlin: Springer) p 103
Molev A I, Nazarov M and Olshanski G I 1996 *Russ. Math. Surv.* **51** 206 (arXiv:hep-th/9409025)
[4] Doikou A 2000 *J. Phys. A: Math. Gen.* **33** 8797 (arXiv:hep-th/0006197)
[5] Molev A I, Ragoucy E and Sorba P 2003 *Rev. Math. Phys.* **15** 789 (arXiv:math/0208140)
Molev A I 2003 *Handbook of Algebra* vol 3, ed M Hazewinkel (Amsterdam: Elsevier) p 907
[6] Baxter R J 1972 *Ann. Phys.* **70** 193
Baxter R J 1973 *J. Stat. Phys.* **8** 25
Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
[7] Takhtajan L A 1990 Quantum groups *Introduction to Quantum Groups and Integrable Massive Models of Quantum Field Theory (Nankai Lectures on Mathematical Physics)* ed M-L Ge and B-H Zhao (Singapore: World Scientific) p 69
[8] Sklyanin E K, Takhtajan L A and Faddeev L D 1979 *Theor. Math. Phys.* **40** 194
[9] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
Jimbo M 1986 *Lett. Math. Phys.* **11** 247
[10] Jimbo M 1986 *Commun. Math. Phys.* **102** 537
[11] Drinfeld V G 1986 *Proc. 1986 Int. Congress of Mathematics, Berkeley* ed A M Gleason (Providence, RI: American Physical Society) p 798
[12] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)
[13] Cherednik I V 1984 *Theor. Math. Phys.* **61** 977
[14] Yang C N and Yang C P 1966 *Phys. Rev.* **150** 327
Yang C N and Yang C P 1969 *J. Math. Phys.* **10** 1115
[15] Zamolodchikov A B and Zamolodchikov Al B 1979 *Ann. Phys.* **120** 253
Faddeev L D 1980 *Sov. Sci. Rev.* C **1** 107
[16] Faddeev L D and Takhtajan L A 1984 *J. Sov. Math.* **24** 241
Faddeev L D and Takhtajan L A 1981 *Phys. Lett.* A **85** 375
[17] Korepin V E 1980 *Theor. Math. Phys.* **76** 165

Korepin V E, Izergin G and Bogoliubov N M 1993 *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz* (Cambridge: Cambridge University Press)

[18] Arnaudon D, Avan J, Crampe N, Doikou A, Frappat L and Ragoucy E 2004 *J. Stat. Mech.* **0408** P005 (arXiv:math-ph/0406021)

[19] Arnaudon D, Crampe N, Doikou A, Frappat L and Ragoucy E 2006 *Int. J. Mod. Phys.* A **21** 1537 (arXiv:math-ph/0503014)

[20] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312

[21] Doikou A 2005 *J. Math. Phys.* **46** 053504 (arXiv:hep-th/0403277)

[22] Crampe N and Doikou A 2007 *J. Math. Phys.* **48** 023511 (arXiv:math-ph/0611030)

[23] Delius G W and Mackay N J 2003 *Commun. Math. Phys.* **233** 173 (arXiv:hep-th/0112023)

[24] Doikou A 2005 *J. Stat. Mech.* P12005 (arXiv:math-ph/0402067)

[25] Doikou A 2005 *Nucl. Phys.* B **725** 493 (arXiv:math-ph/0409060)

[26] Doikou A and Nepomechie R I 1998 *Nucl. Phys.* B **521** 547 (arXiv:hep-th/9803118)

[27] de Vega H J and Gonza'lez Ruiz A 1993 *J. Phys. A: Math. Gen.* **26** L519 (arXiv:hep-th/9211114)

[28] Sfetsos K 1994 *Phys. Lett.* B **324** 335 (arXiv:hep-th/9311010)
Olive D I, Rabinovici E and Schwimmer A 1994 *Phys. Lett.* B **321** 361 (arXiv:hep-th/9311081)
Sfetsos K 1994 *Phys. Rev.* D **50** 2784 (arXiv:hep-th/9402031)

[29] Blau M, Figueroa-O'Farrill J M, Hull C and Papadopoulos G 2002 *Class. Quantum Grav.* **19** L87 (arXiv:hep-th/0201081)

[30] Berenstein D E, Maldacena J M and Nastase H S 2002 *J. High Energy Phys.* JHEP04(2002)013 (arXiv:hep-th/0202021)

[31] Doikou A and Nepomechie R I 1998 *Nucl. Phys.* B **530** 641 (arXiv:hep-th/9807065)

[32] Doikou A 2007 *SIGMA* **3** 009 (arXiv:math-ph/0606040)

[33] Celeghini E, Giachetti R, Sorace E and Tarini M 1990 *J. Math. Phys.* **31** 2548
Schupp P, Watts P and Zumino B 1992 *Lett. Math. Phys.* **24** 141 (arXiv:hep-th/9206024)

[34] Ding J and Frenkel I B 1993 *Commun. Math. Phys.* **156** 277

[35] Rodriguez-Plaza M J 1991 *J. Math. Phys.* **32** 2020
Links J R and Zhang R B 1993 *J. Math. Phys.* **34** 6016
Gould M D and Zhang Y-Z 1994 *Lett. Math. Phys.* **31** 77 (arXiv:hep-th/9304144)

[36] Arnaudon D, Crampe N, Doikou A, Frappat L and Ragoucy E 2006 *Ann. Inst. Henri Poincaré* **7** 1217 (arXiv:math-ph/0512037)

[37] Beisert N 2008 *Adv. Theor. Math. Phys.* **12** 945 (arXiv:hep-th/0511082)

[38] Gomez C and Hernandez R 2006 *J. High Energy Phys.* JHEP11(2006)021 (arXiv:hep-th/0608029)
Gomez C and Hernandez R 2007 *J. High Energy Phys.* JHEP03(2007)108 (arXiv:hep-th/0701200)

[39] Plefka J, Spill F and Torrielli A 2006 *Phys. Rev.* D **74** 066008 (arXiv:hep-th/0608038)